Introduction To Linear Programming

- Today many of the resources needed as inputs to operations are in limited supply.
- Operations managers must understand the impact of this situation on meeting their objectives.
- Linear programming (LP) is one way that operations managers can determine how best to allocate their scarce resources.
- NOTE: Linear Programming is presented in Supplement I for Chapter 11. We will focus on formulation in this class.

Linear Programming (LP) in OM

- There are five common types of decisions in which LP may play a role
  - Product mix
  - Production plan
  - Ingredient mix
  - Transportation
  - Assignment

LP Problems in OM: Product Mix

- <u>Objective</u>

  To select the mix of products or services that results in maximum profits for the planning period

- <u>Decision Variables</u>

  How much to produce and market of each product or service for the planning period

- <u>Constraints</u>

  Maximum amount of each product or service demanded; Minimum amount of product or service policy will allow; Maximum amount of resources available

LP Problems in OM: Production Plan

- Objective

  To select the mix of products or services that results in maximum profits for the planning period

- Decision Variables

  How much to produce on straight-time labor and overtime labor during each month of the year

- Constraints

  Amount of products demanded in each month; Maximum labor and machine capacity available in each month; Maximum inventory space available in each month

Recognizing LP Problems

Characteristics of LP Problems in OM

- A well-defined single objective must be stated.
- There must be alternative courses of action.
- The total achievement of the objective must be constrained by scarce resources or other restraints.
- The objective and each of the constraints must be expressed as linear mathematical functions.

Steps in Formulating LP Problems

1. Define the objective. (min or max)
2. Define the decision variables. (positive, binary)
3. Write the mathematical function for the objective.
4. Write a 1- or 2-word description of each constraint.
5. Write the right-hand side (RHS) of each constraint.
6. Write $\leq$, $=$, or $\geq$ for each constraint.
7. Write the decision variables on LHS of each constraint.
8. Write the coefficient for each decision variable in each constraint.

Example: LP Formulation

Cycle Trends is introducing two new lightweight bicycle frames, the Deluxe and the Professional, to be made from aluminum and steel alloys. The anticipated unit profits are $10 for the Deluxe and $15 for the Professional.

The number of pounds of each alloy needed per frame is summarized on the next slide. A supplier delivers 100 pounds of the aluminum alloy and 80 pounds of the steel alloy weekly. How many Deluxe and Professional frames should Cycle Trends produce each week?

Example: LP Formulation

Pounds of each alloy needed per frame

	Aluminum Alloy	Steel Alloy
Deluxe	2	3
Professional	4	2

Example: LP Formulation

Define the objective

- Maximize total weekly profit

Define the decision variables

- x_1 = number of Deluxe frames produced weekly
- x_2 = number of Professional frames produced weekly

Write the mathematical objective function

- Max $Z = 10x_1 + 15x_2$

Example: LP Formulation

Write a one- or two-word description of each constraint

- Aluminum available
- Steel available

Write the right-hand side of each constraint

- 100
- 80

Write <, =, > for each constraint

- $\leq$ 100
- $\leq$ 80

Example: LP Formulation

- Write all the decision variables on the left-hand side of each constraint
  - $x_1 \quad x_2 \leq 100$
  - $x_1 \quad x_2 \leq 80$

Write the coefficient for each decision in each constraint

  - $+ 2x_1 + 4x_2 \leq 100$
  - $+ 3x_1 + 2x_2 \leq 80$

Example: LP Formulation

- <u>LP in Final Form</u>
  - Max $Z = 10x_1 + 15x_2$
  - Subject To
    - $2x_1 + 4x_2 \leq 100$ (aluminum constraint)
    - $3x_1 + 2x_2 \leq 80$ (steel constraint)
    - $x_1, x_2 \geq 0$ (non-negativity constraints)

Example: LP Formulation

Montana Wood Products manufacturers two-high quality products, tables and chairs. Its profit is \$15 per chair and \$21 per table. Weekly production is constrained by available labor and wood. Each chair requires 4 labor hours and 8 board feet of wood while each table requires 3 labor hours and 12 board feet of wood. Available wood is 2400 board feet and available labor is 920 hours. Management also requires at least 40 tables and at least 4 chairs be produced for every table produced. To maximize profits, how many chairs and tables should be produced?

Example: LP Formulation

Define the objective

- Maximize total weekly profit

Define the decision variables

- x_1 = number of chairs produced weekly
- x_2 = number of tables produced weekly

Write the mathematical objective function

- $\text{Max } Z = 15x_1 + 21x_2$

Example: LP Formulation

Write a one- or two-word description of each constraint

- Labor hours available
- Board feet available
- At least 40 tables
- At least 4 chairs for every table

Write the right-hand side of each constraint

- 920
- 2400
- 40
- 4 to 1 ratio

Write <, =, > for each constraint

- $\leq$ 920
- $\leq$ 2400
- $\geq$ 40
- 4 to 1

Example: LP Formulation

- Write all the decision variables on the left-hand side of each constraint
  - $x_1 \quad x_2 \leq 920$
  - $x_1 \quad x_2 \leq 2400$
  - $x_2 \geq 40$
  - 4 to 1 ratio → $x_1 / x_2 \geq 4/1$

Write the coefficient for each decision in each constraint

  - $+ 4x_1 + 3x_2 \leq 920$
  - $+ 8x_1 + 12x_2 \leq 2400$
  - $x_2 \geq 40$
  - $x_1 \geq 4 x_2$

Example: LP Formulation

- LP in Final Form
  - Max $Z = 15x_1 + 21x_2$
  - Subject To
    - $4x_1 + 3x_2 \leq 920$ (labor constraint)
    - $8x_1 + 12x_2 \leq 2400$ (wood constraint)
    - $x_2 - 40 \geq 0$ (make at least 40 tables)
    - $x_1 - 4x_2 \geq 0$ (at least 4 chairs for every table)
    - $x_1, x_2 \geq 0$ (non-negativity constraints)

Example: LP Formulation

The Sureset Concrete Company produces concrete. Two ingredients in concrete are sand (costs \$6 per ton) and gravel (costs \$8 per ton). Sand and gravel together must make up exactly 75% of the weight of the concrete. Also, no more than 40% of the concrete can be sand and at least 30% of the concrete be gravel. Each day 2000 tons of concrete are produced. To minimize costs, how many tons of gravel and sand should be purchased each day?

Example: LP Formulation

Define the objective

- Minimize daily costs

Define the decision variables

- x_1 = tons of sand purchased
- x_2 = tons of gravel purchased

Write the mathematical objective function

- Min $Z = 6x_1 + 8x_2$

Example: LP Formulation

Write a one- or two-word description of each constraint

- 75% must be sand and gravel
- No more than 40% must be sand
- At least 30% must be gravel

Write the right-hand side of each constraint

- .75(2000)
- .40(2000)
- .30(2000)

Write <, =, > for each constraint

- = 1500
- $\leq$ 800
- $\geq$ 600

Example: LP Formulation

- Write all the decision variables on the left-hand side of each constraint
  - $x_1 \quad x_2 = 1500$
  - $x_1 \leq 800$
  - $x_2 \geq 600$

Write the coefficient for each decision in each constraint

- $+ x_1 + x_2 = 1500$
- $+ x_1 \leq 800$
- $x_2 \geq 600$

Example: LP Formulation

- LP in Final Form
  - Min $Z = 6x_1 + 8x_2$
  - Subject To
    - $x_1 + x_2 = 1500$ (mix constraint)
    - $x_1 \leq 800$ (mix constraint)
    - $x_2 \geq 600$ (mix constraint)
    - $x_1, x_2 \geq 0$ (non-negativity constraints)

LP Problems in General

- Units of each term in a constraint must be the same as the RHS
- Units of each term in the objective function must be the same as Z
- Units between constraints do not have to be the same
- LP problem can have a mixture of constraint types

LP Problem

- Galaxy Ind. produces two water guns, the Space Ray and the Zapper. Galaxy earns a profit of \$3 for every Space Ray and \$2 for every Zapper. Space Rays and Zappers require 2 and 4 production minutes per unit, respectively. Also, Space Rays and Zappers require .5 and .3 pounds of plastic, respectively. Given constraints of 40 production hours, 1200 pounds of plastic, Space Ray production can't exceed Zapper production by more than 450 units; formulate the problem such that Galaxy maximizes profit.

LP Model

R = # of Space Rays to produce

Z = # of Zappers to produce

Max $Z = 3.00R + 2.00Z$

ST

$2R + 4Z \leq 2400$	can't exceed available hours (40*60)
$.5R + .3Z \leq 1200$	can't exceed available plastic
$R - S \leq 450$	Space Rays can't exceed Zappers by more than 450
$R, S \geq 0$	non-negativity constraint

LP Problem

The White Horse Apple Products Company purchases apples from local growers and makes applesauce and apple juice. It costs $0.60 to produce a jar of applesauce and $0.85 to produce a bottle of apple juice. The company has a policy that at least 30% but not more than 60% of its output must be applesauce.

The company wants to meet but not exceed demand for each product. The marketing manager estimates that the maximum demand for applesauce is 5,000 jars, plus an additional 3 jars for each $1 spent on advertising. Maximum demand for apple juice is estimated to be 4,000 bottles, plus an additional 5 bottles for every $1 spent to promote apple juice. The company has $16,000 to spend on producing and advertising applesauce and apple juice. Applesauce sells for $1.45 per jar; apple juice sells for $1.75 per bottle. The company wants to know how many units of each to produce and how much advertising to spend on each in order to maximize profit.

LP Model

S = # jars apple Sauce to make

J = # bottles apple Juice to make

SA = $ for apple Sauce Advertising

JA = $ for apple Juice Advertising

Max $Z = 1.45S + 1.75J - .6S - .85J - SA - JA$

ST

$S \geq .3(S + J)$	at least 30% apple sauce
$S \leq .6(S + J)$	no more than 60% apple sauce
$S \leq 5000 + 3SA$	don't exceed demand for apple sauce
$J \leq 4000 + 5JA$	don't exceed demand for apple juice
$.6S + .85J + SA + JA \leq 16000$	budget

LP Problem

A ship has two cargo holds, one fore and one aft. The fore cargo hold has a weight capacity of 70,000 pounds and a volume capacity of 30,000 cubic feet. The aft hold has a weight capacity of 90,000 pounds and a volume capacity of 40,000 cubic feet. The shipowner has contracted to carry loads of packaged beef and grain. The total weight of the available beef is 85,000 pounds; the total weight of the available grain is 100,000 pounds. The volume per mass of the beef is 0.2 cubic foot per pound, and the volume per mass of the grain is 0.4 cubic foot per pound. The profit for shipping beef is $0.35 per pound, and the profit for shipping grain is $0.12 per pound. The shipowner is free to accept all or part of the available cargo; he wants to know how much meat and grain to accept in order to maximize profit.

LP Model

BF = # lbs beef to load in fore cargo hold

BA = # lbs beef to load in aft cargo hold

GF = # lbs grain to load in fore cargo hold

GA = # lbs grain to load in aft cargo hold

Max Z = .35 BF + .35BA + .12GF + .12 GA

ST

BF + GF ≤ 70000 fore weight capacity – lbs

BA + GA ≤ 90000 aft weight capacity – lbs

.2BF + .4GF ≤ 30000 for volume capacity – cubic feet

.2BA + .4GA ≤ 40000 for volume capacity – cubic feet

BF + BA ≤ 85000 max beef available

GF + GA ≤ 100000 max grain available

LP Problem

In the summer, the City of Sunset Beach staffs lifeguard stations seven days a week. Regulations require that city employees (including lifeguards) work five days a week and be given two consecutive days off. Insurance requirements mandate that Sunset Beach provide at least one lifeguard per 8000 average daily attendance on any given day. The average daily attendance figures by day are as follows: Sunday – 58,000, Monday – 42,000, Tuesday – 35,000, Wednesday – 25,000, Thursday – 44,000, Friday – 51,000 and Saturday – 68,000. Given a tight budget constraint, the city would like to determine a schedule that will employ as few lifeguards as possible.

LP Model

- X1 = number of lifeguards scheduled to begin on Sunday
- X2 = “ “ “ “ “ Monday
- X3 = “ “ “ “ “ Tuesday
- X4 = “ “ “ “ “ Wednesday
- X5 = “ “ “ “ “ Thursday
- X6 = “ “ “ “ “ Friday
- X7 = “ “ “ “ “ Saturday

LP Model

Min X1 + X2 + X3 + X4 + X5 + X6 + X7

ST

X1 + X4 + X5 + X6 +X7 ≥ 8 (Sunday)

X1 + X2 + X5 + X6 +X7 ≥ 6 (Monday)

X1 + X2 + X3 + X6 +X7 ≥ 5 (Tuesday)

X1 + X2 + X3 + X4 +X7 ≥ 4 (Wednesday)

X1 + X2 + X3 + X4 +X5 ≥ 6 (Thursday)

X2 + X3 + X4 + X5 +X6 ≥ 7 (Friday)

X3 + X4 + X5 + X6 +X7 ≥ 9 (Saturday)

All variables ≥ 0 and integer