Fourier Transforms

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Fourier Transform

Outline of the Talk

What is the Fourier Transform?

Fourier Cosine Series for even functions

Fourier Sine Series for odd functions

The continuous limit: the Fourier transform (and its inverse)

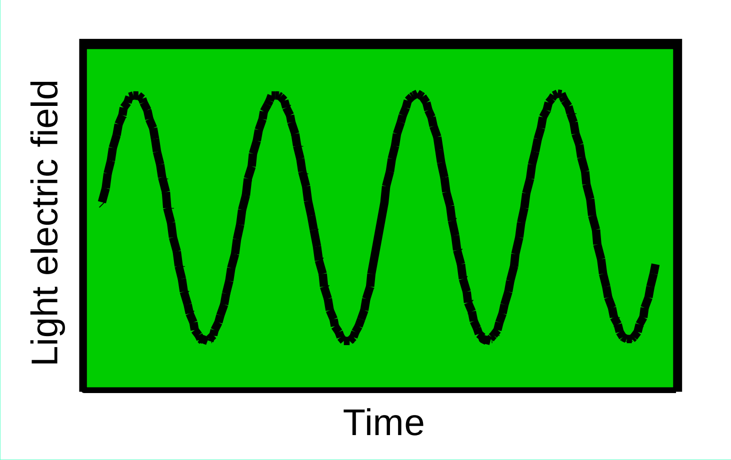
Some transform examples and the Dirac delta function

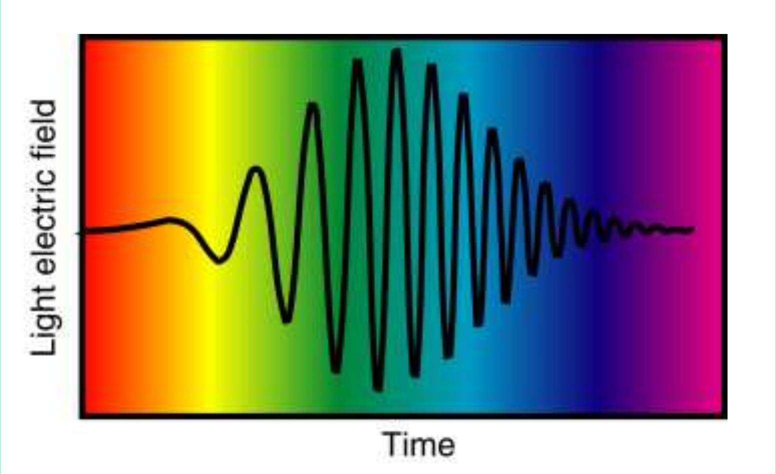
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp(i\omega t)\, d\omega \qquad F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\, dt$$

What do we hope to achieve with the Fourier Transform?

We desire a measure of the frequencies present in a wave. This will lead to a definition of the term, the **spectrum**.

Plane waves have only one frequency, ω.



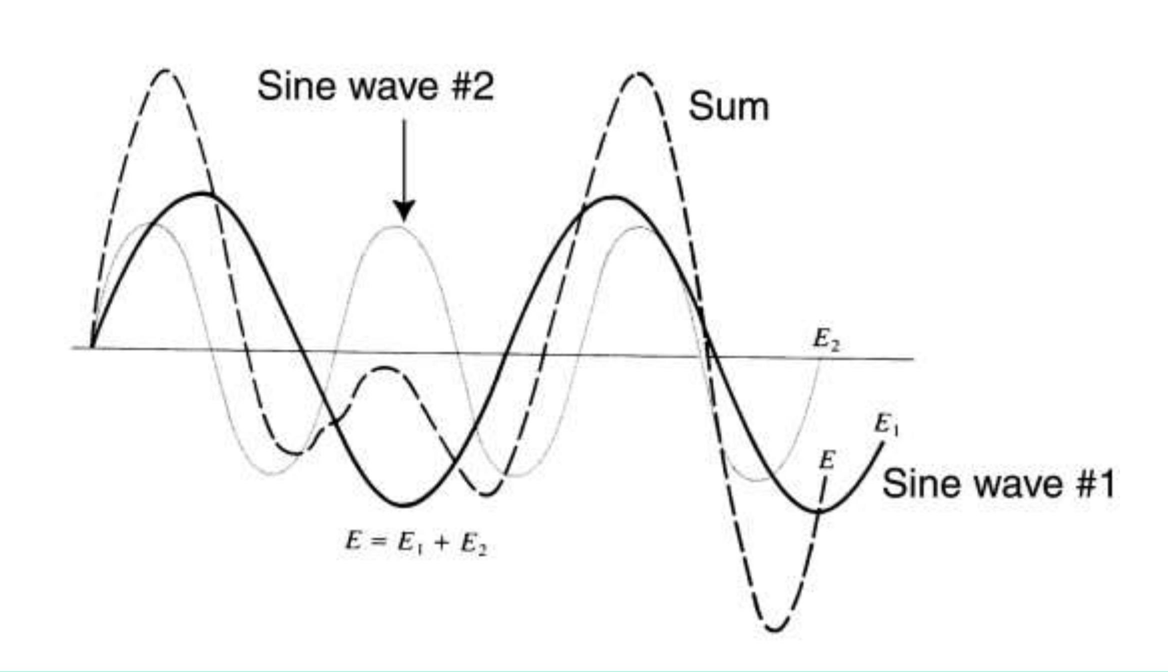


This light wave has many frequencies. And the frequency increases in time (from red to blue).

It will be nice if our measure also tells us **when** each frequency occurs.

Anharmonic waves are sums of sinusoids.

Consider the sum of two sine waves (i.e., harmonic waves) of different frequencies:



The resulting wave is periodic, but not harmonic.
Essentially all waves are anharmonic.

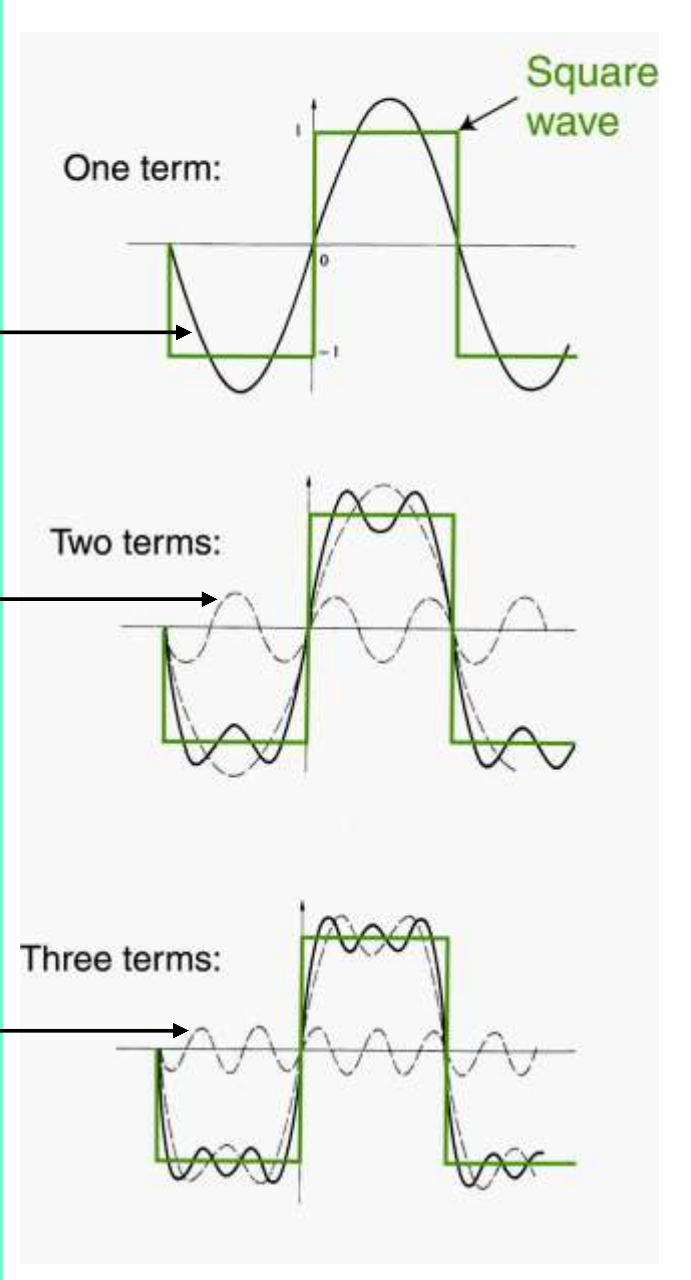
Fourier decomposing functions

Here, we write a **square wave** as a sum of sine waves.

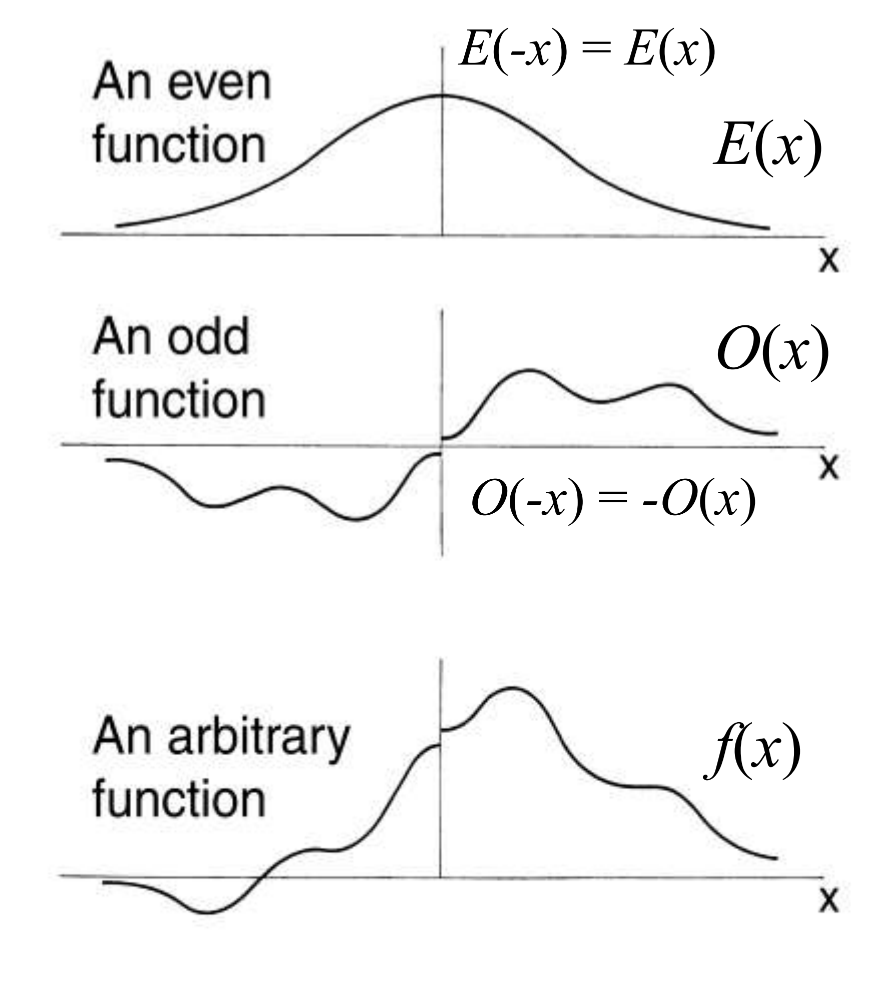
$\sin(\omega t)$

$\sin(3\omega t)$

$\sin(5\omega t)$



Any function can be written as the sum of an even and an odd function.



$$E(x) \equiv [\,f(x) + f(-x)]/2$$

$$O(x) \equiv [\,f(x) - f(-x)]/2$$

$$\Downarrow$$

$$f(x) = E(x) + O(x)$$

The Fourier Transform

Consider the Fourier coefficients. Let's define a function $F(m)$ that incorporates both cosine and sine series coefficients, with the sine series distinguished by making it the imaginary component:

$$F(m) \equiv F_m - i\,F'_m = \int f(t)\cos(mt)\,dt \;-\; i\int f(t)\sin(mt)\,dt$$

Let's now allow $f(t)$ to range from $-\infty$ to ∞, so we'll have to integrate from $-\infty$ to ∞, and let's redefine m to be the "frequency," which we'll now call ω:

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\,dt}$$

The Fourier Transform

$F(\omega)$ is called the Fourier Transform of $f(t)$. **It contains equivalent information to that in $f(t)$.** We say that $f(t)$ lives in the **time domain**, and $F(\omega)$ lives in the **frequency domain**. $F(\omega)$ is just another way of looking at a function or wave.

The Inverse Fourier Transform

The Fourier Transform takes us from $f(t)$ to $F(\omega)$.
How about going back?

Recall our formula for the Fourier Series of $f(t)$:

$$f(t) = \frac{1}{\pi}\sum_{m=0}^{\infty} F_m \cos(mt) + \frac{1}{\pi}\sum_{m=0}^{\infty} F_m' \sin(mt)$$

Now transform the sums to integrals from –∞ to ∞, and again replace F_m with $F(\omega)$. Remembering the fact that we introduced a factor of i (and including a factor of 2 that just crops up), we have:

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp(i\omega t)\,d\omega}$$

Inverse Fourier Transform

The Fourier Transform and its Inverse

The Fourier Transform and its Inverse:

$$\boxed{F(\omega) \;=\; \int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\,dt}$$

FourierTransform

$$\boxed{f(t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp(i\omega t)\,d\omega}$$

Inverse Fourier Transform

So we can transform to the frequency domain and back. Interestingly, these transformations are very similar.

There are different definitions of these transforms. The 2π can occur in several places, but the idea is generally the same.

Fourier Transform Notation

There are several ways to denote the Fourier transform of a function.

If the function is labeled by a lower-case letter, such as f, we can write:

$$f(t) \rightarrow F(\omega)$$

If the function is already labeled by an upper-case letter, such as E, we can write:

$$E(t) \rightarrow \mathscr{F}\{E(t)\} \quad \text{or:} \quad E(t) \rightarrow \tilde{E}(\omega)$$

Sometimes, this symbol is used instead of the arrow: $\supset$

The Spectrum

We define the spectrum, $S(\omega)$, of a wave $E(t)$ to be:

$$\boxed{S(\omega) \equiv \left| \mathcal{F}\{E(t)\} \right|^2}$$

This is the measure of the frequencies present in a light wave.

Example: Fourier Transform of a rectangle function: rect(t)

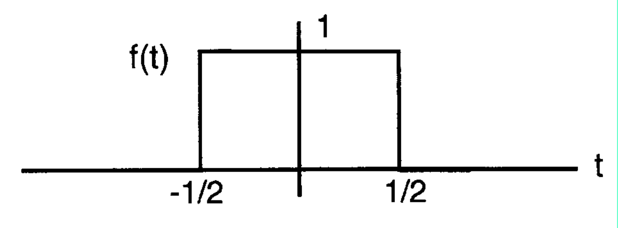
$$F(\omega) = \int_{-1/2}^{1/2} \exp(-i\omega t)\,dt = \frac{1}{-i\omega}\left[\exp(-i\omega t)\right]_{-1/2}^{1/2}$$

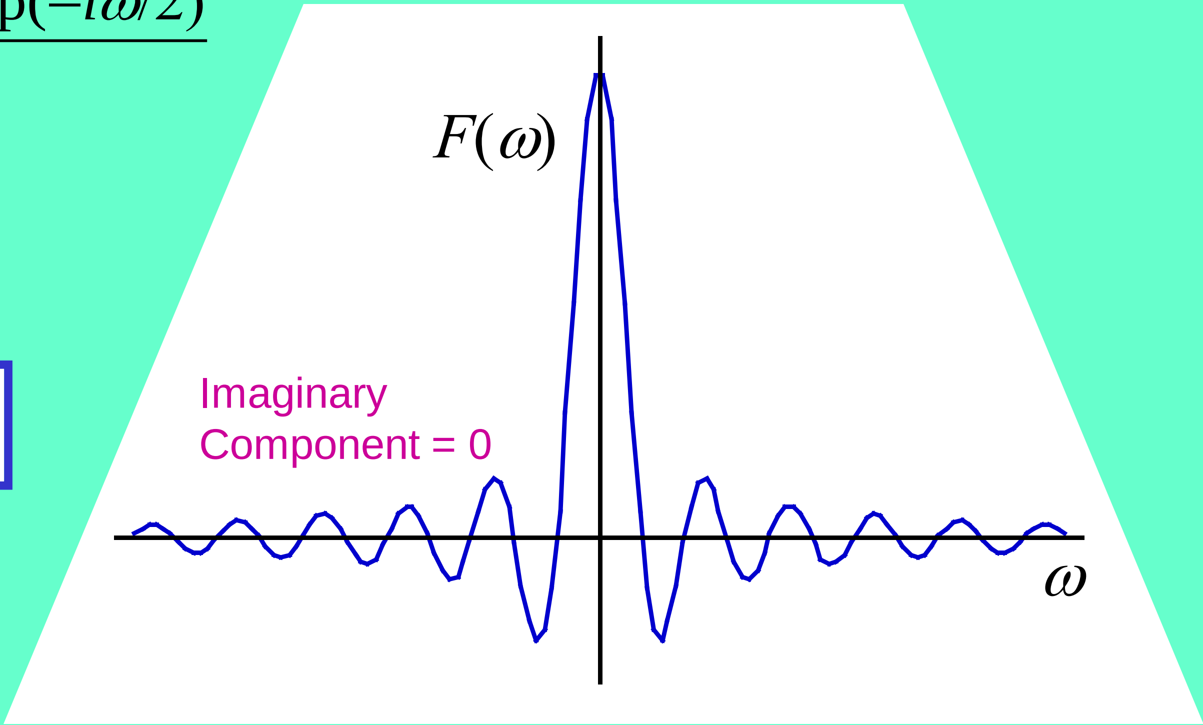
$$= \frac{1}{-i\omega}\left[\exp(-i\omega/2) - \exp(i\omega/2)\right]$$

$$= \frac{1}{(\omega/2)} \frac{\exp(i\omega/2) - \exp(-i\omega/2)}{2i}$$

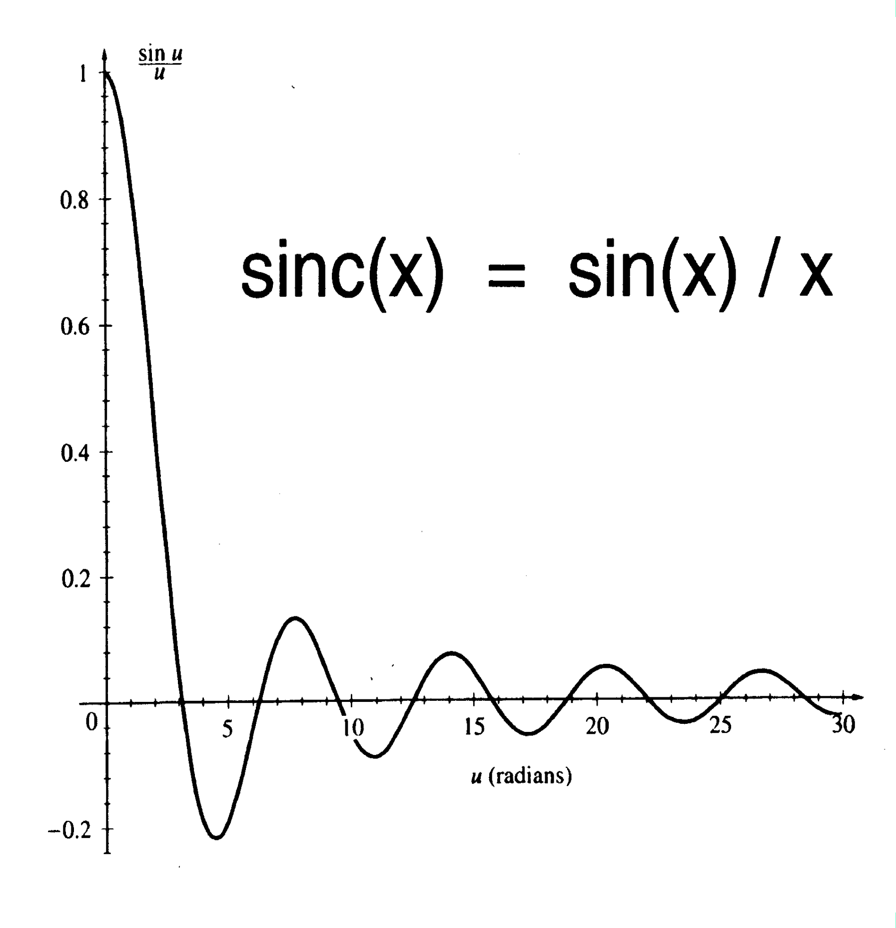
$$= \frac{\sin(\omega/2)}{(\omega/2)}$$

$$\boxed{F(\omega) = \mathrm{sinc}(\omega/2)}$$





Sinc(x) and why it's Important



Sinc(x/2) is the Fourier transform of a rectangle function.

Sinc^2(x/2) is the Fourier transform of a triangle function.

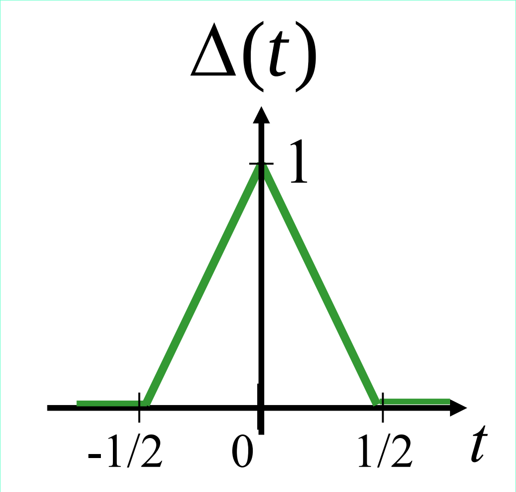
Sinc^2(ax) is the diffraction pattern from a slit.

It just crops up everywhere...

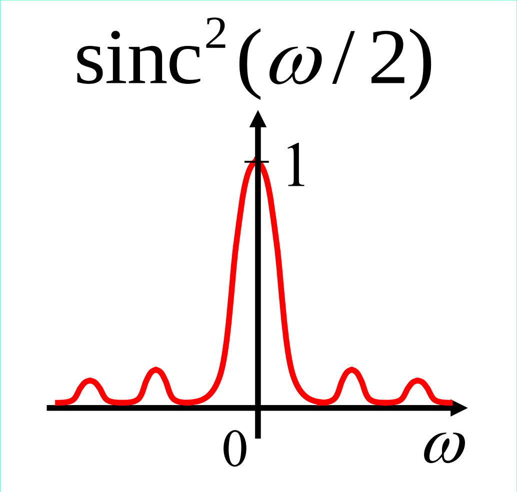
The Fourier Transform of the triangle function, $\Delta(t)$, is $\mathrm{sinc}^2(\omega/2)$

The triangle function is just what it sounds like.

Sometimes people use $\Lambda(t)$, too, for the triangle function.



$\supset$



We'll prove this when we learn about convolution.

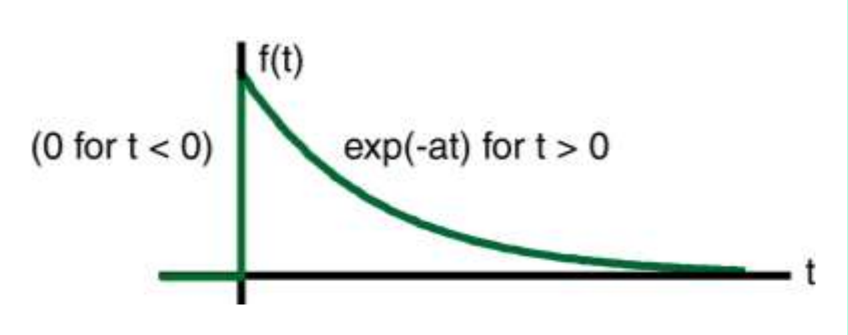
Example: Fourier Transform of a decaying exponential: exp(-*at*) (*t* > 0)

$$F(\omega) = \int_0^\infty \exp(-at)\exp(-i\omega t)\,dt$$

$$= \int_0^\infty \exp(-at - i\omega t)\,dt = \int_0^\infty \exp(-[a+i\omega]t)\,dt$$

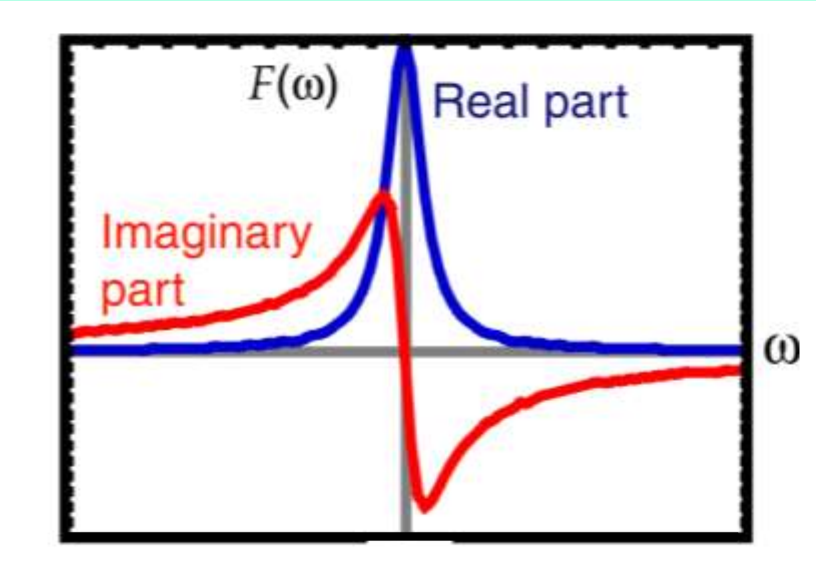
$$= \frac{-1}{a+i\omega}\exp(-[a+i\omega]t)\Big|_0^{+\infty} = \frac{-1}{a+i\omega}[\exp(-\infty) - \exp(0)]$$

$$= \frac{-1}{a+i\omega}[0-1] = \frac{1}{a+i\omega}$$



$$\boxed{F(\omega) = -i\frac{1}{\omega - ia}}$$

A complex Lorentzian!

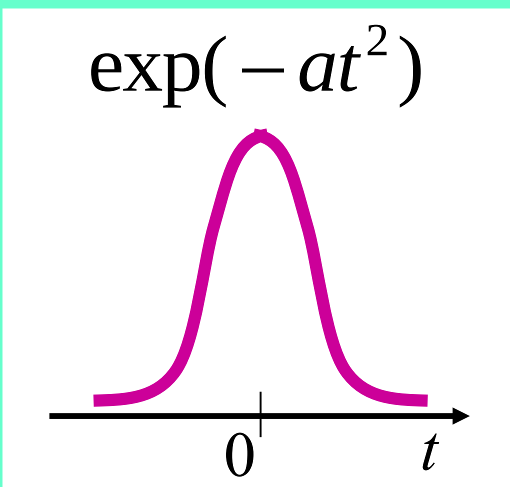


Example: Fourier Transform of a Gaussian, $\exp(-at^2)$, is itself!

$$\mathscr{F}\{\exp(-at^2)\} = \int_{-\infty}^{\infty} \exp(-at^2)\exp(-i\omega t)\,dt$$

$$\boxed{\propto \exp(-\omega^2/4a)}$$

The details are a HW problem!



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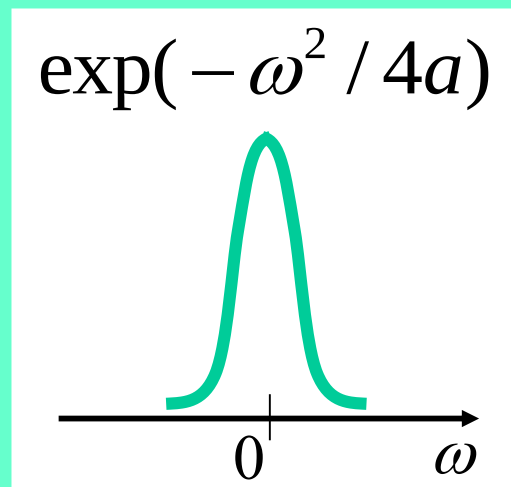
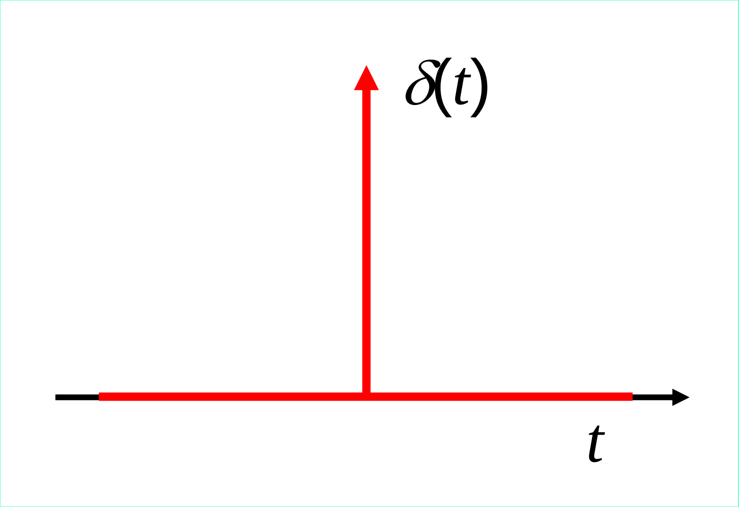


The Dirac delta function

Unlike the Kronecker delta-function, which is a function of two integers, the Dirac delta function is a function of a real variable, t.

$$\delta(t) \equiv \begin{cases} \infty \text{ if } t = 0 \\ 0 \text{ if } t \neq 0 \end{cases}$$

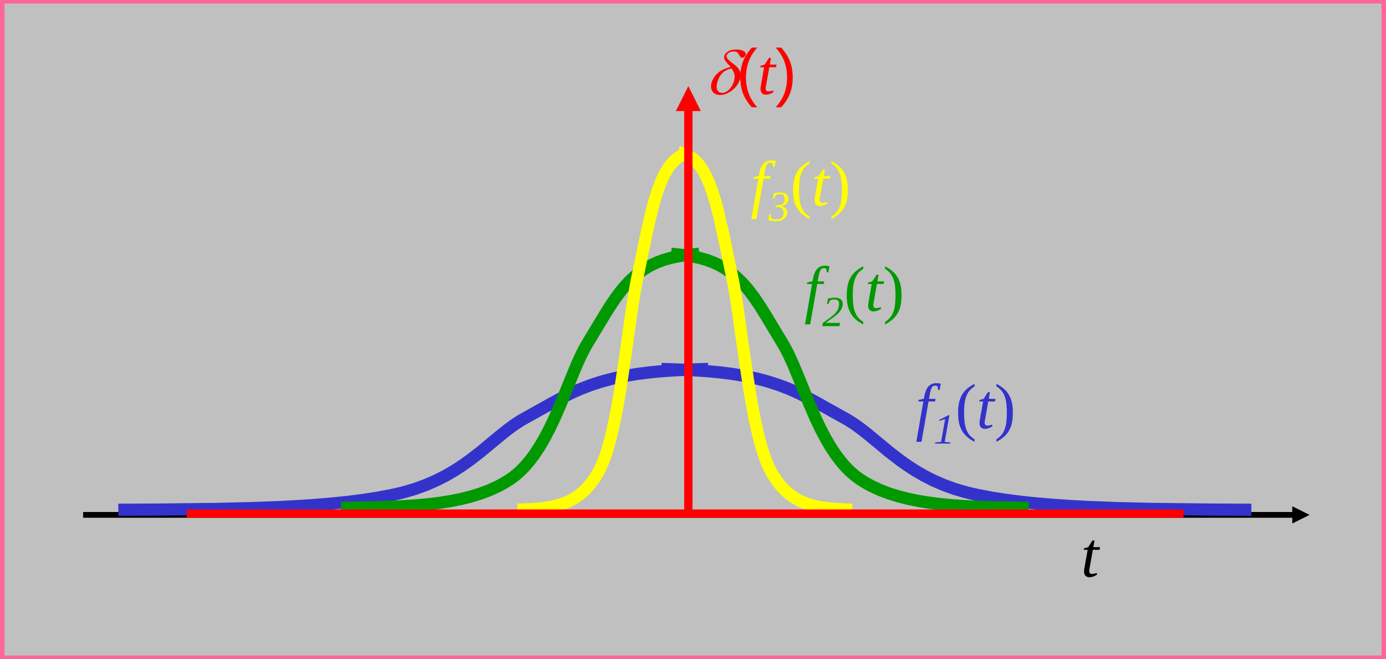


The Dirac delta function

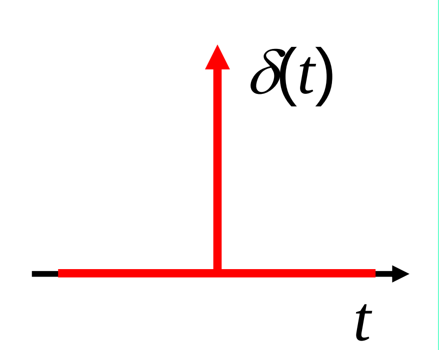
$$\delta(t) \equiv \begin{cases} \infty \text{ if } t = 0 \\ 0 \text{ if } t \neq 0 \end{cases}$$

It's best to think of the delta function as the limit of a series of peaked continuous functions.

$$f_m(t) = m \exp[-(mt)^2]/\sqrt{\pi}$$



Dirac δ–function Properties



$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

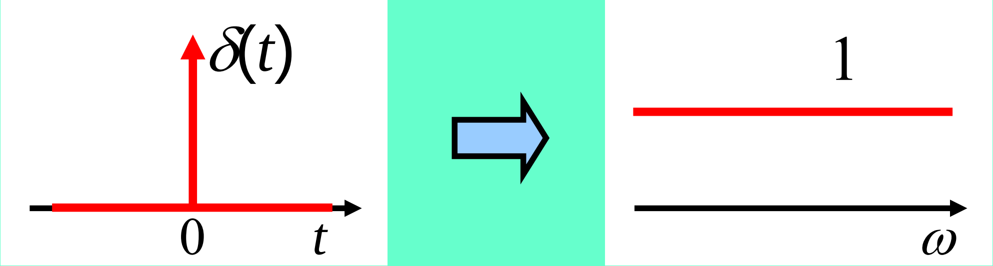
$$\int_{-\infty}^{\infty} \delta(t-a)\, f(t)\, dt = \int_{-\infty}^{\infty} \delta(t-a)\, f(a)\, dt = f(a)$$

$$\int_{-\infty}^{\infty} \exp(\pm i\omega t)\, dt = 2\pi\, \delta(\omega)$$

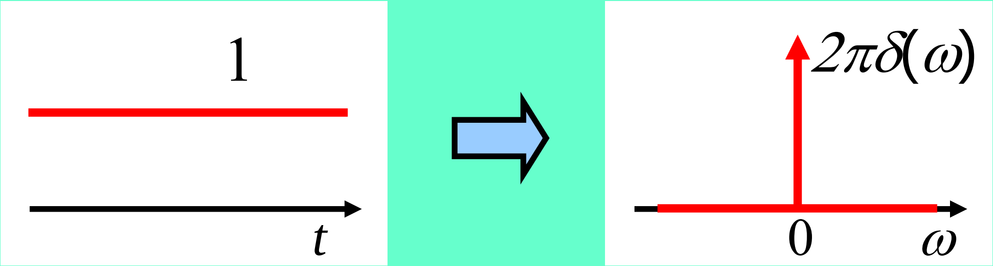
$$\int_{-\infty}^{\infty} \exp[\pm i(\omega-\omega')t]\ dt = 2\pi\, \delta(\omega-\omega')$$

The Fourier Transform of $\delta(t)$ is 1

$$\int_{-\infty}^{\infty} \delta(t)\exp(-i\omega t)\,dt = \exp(-i\omega[0]) = 1$$



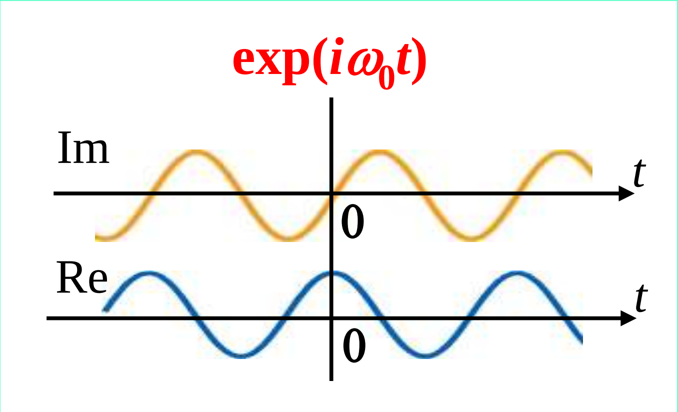
And the Fourier Transform of 1 is $2\pi\delta(\omega)$: $\displaystyle\int_{-\infty}^{\infty} 1\exp(-i\omega t)\,dt = 2\pi\,\delta(\omega)$

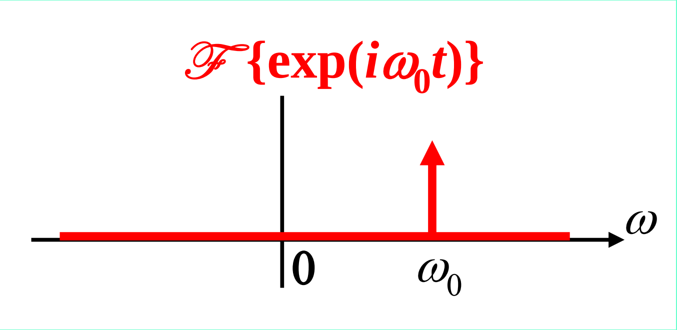


The Fourier transform of $\exp(i\omega_0 t)$

$$\mathscr{F}\{\exp(i\omega_0 t)\} = \int_{-\infty}^{\infty} \exp(i\omega_0 t)\exp(-i\omega t)\,dt$$

$$= \int_{-\infty}^{\infty} \exp(-i[\omega-\omega_0]t)\,dt = 2\pi\,\delta(\omega-\omega_0)$$





The function $\exp(i\omega_0 t)$ is the essential component of Fourier analysis.

It is a pure frequency.

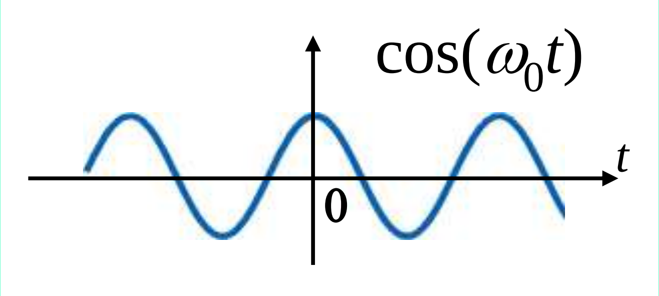
The Fourier transform of $\cos(\omega_0\, t)$

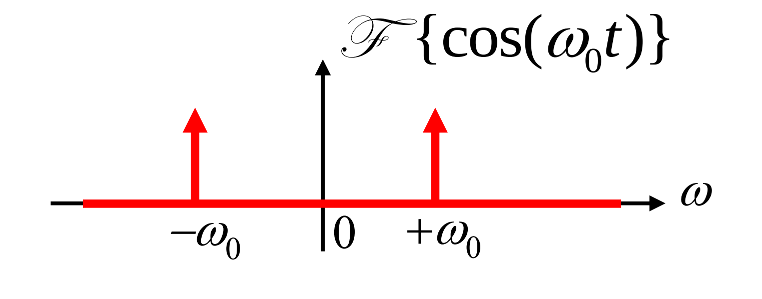
$$\mathscr{F}\{\cos(\omega_0 t)\} = \int_{-\infty}^{\infty} \cos(\omega_0 t)\exp(-i\,\omega t)\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \left[\exp(i\,\omega_0\, t) + \exp(-i\,\omega_0\, t)\right]\exp(-i\,\omega t)\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \exp(-i[\omega-\omega_0]t)\,dt \quad + \quad \frac{1}{2}\int_{-\infty}^{\infty} \exp(-i[\omega+\omega_0]t)\,dt$$

$$= \pi\,\delta(\omega-\omega_0) \quad + \quad \pi\,\delta(\omega+\omega_0)$$





Fourier Transform Symmetry Properties

Expanding the Fourier transform of a function, $f(t)$:

$$F(\omega) = \int_{-\infty}^{\infty} [\mathrm{Re}\{f(t)\} + i\,\mathrm{Im}\{f(t)\}]\,[\cos(\omega t) - i\sin(\omega t)]\,dt$$

Expanding more, noting that: $\int_{-\infty}^{\infty} O(t)\,dt = 0$ if $O(t)$ is an odd function

= 0 if $\mathrm{Re}\{f(t)\}$ is odd ↓ = 0 if $\mathrm{Im}\{f(t)\}$ is even ↓

$$F(\omega) = \int_{-\infty}^{\infty} \mathrm{Re}\{f(t)\}\cos(\omega t)\,dt + \int_{-\infty}^{\infty} \mathrm{Im}\{f(t)\}\sin(\omega t)\,dt \quad \leftarrow \mathrm{Re}\{F(\omega)\}$$

= 0 if $\mathrm{Im}\{f(t)\}$ is odd ↓ = 0 if $\mathrm{Re}\{f(t)\}$ is even ↓

$$+\; i\int_{-\infty}^{\infty} \mathrm{Im}\{f(t)\}\cos(\omega t)\,dt - i\int_{-\infty}^{\infty} \mathrm{Re}\{f(t)\}\sin(\omega t)\,dt \quad \leftarrow \mathrm{Im}\{F(\omega)\}$$

Even functions of ω **Odd functions of ω**

Some functions don't have Fourier Transforms.

The condition for the existence of a given $F(\omega)$ is:

$$\int_{-\infty}^{\infty} \left| f(t) \right| \, dt \;<\; \infty$$

Functions that do not asymptote to zero in both the $+\infty$ and $-\infty$ directions generally do not have Fourier transforms.

So we'll assume that all functions of interest go to zero at $\pm\infty$.

THANKS