Classification of Groups of Order 24

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Theorems

- The Sylow Theorems:
- Sylow **Theorem 1**: There exists a p-sylow subgroup of *G*, of order p^n, where p^n divides the order of G but p^{n+1} does not.
- **Corollary**: Given a finite group *G* and a prime number *p* dividing the order of *G*, then there exists an element of order *p* in *G* .
- **Sylow Theorem 2**: All Sylow *p*-subgroups of *G* are conjugate to each other (and therefore isomorphic), i.e. if *H* and *K* are p-sylow subgroup of *G*, then there exists an element *g* in *G* with $g^{-1}Hg = K$.
- **Sylow Theorem 3**: Let n_p be the number of Sylow *p*-subgroups of *G*.

Then $n_p = 1 \bmod p$.

More Definitions and Theorems

- Thm: If H_1 and H_2 are subgroups a group G, and $P = H_1 \bigcap H_2$ then we have the following:

$$|P| \geq \frac{|H_1||H_2|}{|G|}$$

Theorem: Suppose H and K are subgroups, where H is normal, of a group G and H intersect K only contains the identity. Then o(HK)=o(G) and G is isomorphic to the semidirect product of H and K.

More Theorems

Theorem: Two different actions of H on K make two different Semidirect Product structures.

Note: An action of a group H on a set X is a homomorphism from H $\rightarrow$ Sym(X). In the case where X is a group, Sym(X)=Aut(X)

Theorem: Two conjugate automorphisms spawn the same (isomorphic) semidirect product structures.

Theorems(Cont.)

- If H is the only subgroup of order n of a group G, then H is normal in G.
- If the index of a subgroup H of a group G is 2, then H is normal in G.

Preliminaries

- Let G be a group of order 24. The only distinct prime factors of 24 are 2 and 3.
- So, we have a 2-sylow subgroup, H, of order 8 and a 3-sylow subgroup, K, of order 3.
- But the problem is that neither H nor K is necessarily normal in G, so we cannot invoke the Semidirect Product Theorem.
- So we have three cases.

Cases

- Case 1: H is normal in G.
- Case 2: K is normal in G.
- Case 3: Neither H nor K is normal in G.

More about H and K

- K is of order 3; therefore, it is isomorphic to Z_3.
- H, unfortunately, can be any of the following groups:

A. Z_8
B. $Z_4 X Z_2$
C. $Z_2 X Z_2 X Z_2$
D. D_4
E. $Q_{(8)}$ (The Quaternion Group)

Both of the non-identity elements of K has order 3. On the other hand, non of the elements of H has order 3. Therefore, H and K must be non-trivially disjoint.

Case 1, H is Normal

We can find a non-trivial action of H on K if and only if we can find a homomorphism from K into Aut(H). Since homomorphisms conserve the order of elements, Aut(H) must have some elements of order 3 for there to be a non-trivial semidirect product.

- H is isomorphic to Z_8

The automorphism of H has 4 elements.. But no group of order 4 has any element of order 3. Therefore, there can be no homomorphism between K and Aut(H).

So, we can make no non-trivial semidirect products of H and K, yielding the direct product Z_{24}.

B. H is isomorphic to Z_4 X Z_2

The Aut(H) is the dihedral group of 8 elements, which has no elements of order 3.

As before, we only have the direct product Z_{12} X Z_2.

Case 1(cont.)

C. H is isomorphic to Z_2 X Z_2 X Z_2

Aut(H)= GLN(3, Z_3), which has 54 elements of order 3. But all of these elements are conjugate, so we only get one non-trivial semidirect product, as well as the trivial direct product.

D. H is isomorphic to D_4

Interestingly, Aut(H)=D_4. Again, D_4 has no elements of order 3, so we have the trivial direct product: D_4 X Z_3

E. H is isomorphic to $Q_{(8)}$

Aut(H)= S_4, which has 8 elements of order 3. But again, they are all conjugate so we only get one non-trivial semidirect product: $Q_{(8)}$XI Z_3

And the trivial direct product: $Q_{(8)}$X Z_3.

Case 2, K is Normal

- This part is very similar to the case where H is normal, and we actually get no new groups from this case.

Case 3, Neither H nor K is Normal

- By the sylow theorems, the number of subgroups of order 8 of G must be 1 mod 2. Since H is not normal, the number of 2-sylow subgroups is greater than or equal to 3.

- We will now take two of the 2-sylow subgroups and call them H_1 and H_2. Let

$$P = H_1 \bigcap H_2$$

- Then

$$|P| \geq \frac{|H_1||H_2|}{|G|}$$

So, o(P)>2. Since P is a subgroup of H_1 and H_2 its order must divide 8. Therefore, o(P)=4. But since the index of P in H_1 and H_2 is 2, P is normal in H_1 and H_2. Thus its normalizer, N_p, is at least of order 8+4=12.

Thus, either P is normal in a subgroup of order 12, or P is normal in G.

Now take K=<s> to be any 3-sylow subgroup of G. Let M=PK. Then M is of order 12. The index of M in G is 2, so M is normal in G.

Subcase 1

- H is isomorphic to Z_8

Since H is cyclic, it can be generated by some element p in G, so H=<p>. Then P=<p^2> is isomorphic to Z_4. So $M \cong Z_4 X Z_3$

Since K is the only subgroup of order 3 in M, K is normal in M.

Further, since conjugation (a group automorphism) conserves the order of elements, $o(s)=o(psp^{-1})$. So, $psp^{-1}=s$ or $psp^{-1}=s^{-1}$. If the former were true, then we would get Case 2 again, so

$psp^{-1}=s^{-1}$.

Now we check that H is not normal in G:

$sps^{-1}=s^{-1}p$. So, H cannot be normal in G because conjugation of p by s yields an element of G not in H. This gives us another possible group structure.

Subcase 2

- H is isomorphic to Z_4 X Z_2

  So, $H=<p,t\ I\ p^4=t^2=e,\ pt=tp>$

A. P is isomorphic to Z_4. Then $P=<p>$.

So, M is isomorphic to $Z_4 X Z_3=<p,s\ I\ p^4=s^3=e,\ ps=sp>$.

Again, tst=s or $tst=s^{-1}$. And as before, the latter must be true. As in the previous subcase, H is not normal in G, so we get another group of order 24.

B. P is isomorphic to Z_2 X Z_2

We have two cases:

1. M is isomorphic to Z_2 X Z_2 X Z_3
2. M is isomorphic to Z_2 X Z_2 XI Z_3

Subcase 2(cont.)

- 1. M is isomorphic to Z_2 X Z_2 X Z_3.

M is abelian; so we have the following:

$p^2s=sp^2$, ts=st, pt=tp.

As before, $psp^{-1}=s^{-1}$. So, $sps^{-1}=s^{-1}p$. So, H and K are not normal in G, and we get another group of order 24.

- 2. M is isomorphic to Z_2 X Z_2 XI Z_3.

So, we have the following:

$sp^2=ts$, $st=p^2ts$, pt=tp

But we notice that all of the elements of M have order 3, except those in P. So suppose $psp^{-1}=p^{2k}t^ls$.

Then $p^2sp^2=pp^{2k}t^lsp^{-1}=p^{2k}t^lpsp^{-1}=p^{2k}t^lp^{2k}t^ls=s$. But this would imply that $p^2t=e$.

This is a contradiction.

Thus no group can be formed this way.

Subcases 3, 4, and 5

- H is isomorphic to $Z_2 \times Z_2 \times Z_2$: we get 2 more groups.
- H is isomorphic to D_4: we get 2 more groups.
- H is isomorphic to $Q_{(8)}$: we get 1 more group.
- Showing the above follows a similar procedure as Subcases 1 and 2.
- This concludes the classification of groups of order 24 and we get 15 groups (3 abelian and 12 non-abelian).